Exercise 3.4.11

Consider the *nonhomogeneous* heat equation (with a steady heat source):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x).$$

Solve this equation with the initial condition

$$u(x, 0) = f(x)$$

and the boundary conditions

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.$$

Assume that a continuous solution exists (with continuous derivatives). [*Hints*: Expand the solution as a Fourier sine series (i.e., use the method of eigenfunction expansion). Expand $g(x)$ as a Fourier sine series. Solve for the Fourier sine series of the solution. Justify all differentiations with respect to x.]

Solution

In order for the homogeneous Dirichlet boundary conditions to be satisfied, we assume the solution has the form of a Fourier sine series.

$$u(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{n\pi x}{L} \tag{1}$$

Since u is continuous, this is justified. Apply the initial condition now to determine $B_n(0)$.

$$u(x, 0) = \sum_{n=1}^{\infty} B_n(0) \sin\frac{n\pi x}{L} = f(x) \quad \rightarrow \quad B_n(0) = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

This formula for $B_n(0)$ will be needed later. Assuming $\partial u/\partial t$ is continuous, term-by-term differentiation with respect to t is valid.

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} B_n'(t) \sin\frac{n\pi x}{L}$$

Because u is continuous and $u(0, t) = u(L, t) = 0$, differentiation of the sine series with respect to x is valid.

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n(t) \cos\frac{n\pi x}{L}$$

$\partial u/\partial x$ is continuous, so differentiation of this cosine series with respect to x is valid.

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \left(-\frac{n^2\pi^2}{L^2}\right) B_n(t) \sin\frac{n\pi x}{L}$$

Substitute these infinite series into the PDE.

$$\sum_{n=1}^{\infty} B_n'(t) \sin\frac{n\pi x}{L} = k\sum_{n=1}^{\infty} \left(-\frac{n^2\pi^2}{L^2}\right) B_n(t) \sin\frac{n\pi x}{L} + g(x)$$

Bring them both to the left side.

$$\sum_{n=1}^{\infty} B_n'(t) \sin \frac{n\pi x}{L} + k \sum_{n=1}^{\infty} \left(\frac{n^2\pi^2}{L^2}\right) B_n(t) \sin \frac{n\pi x}{L} = g(x)$$

Combine them and factor the summand.

$$\sum_{n=1}^{\infty} \left[B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t)\right] \sin \frac{n\pi x}{L} = g(x)$$

To obtain the quantity in square brackets, multiply both sides by $\sin \frac{p\pi x}{L}$, where p is an integer,

$$\sum_{n=1}^{\infty} \left[B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t)\right] \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L} = g(x) \sin \frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} \left[B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t)\right] \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}\, dx = \int_0^L g(x) \sin \frac{p\pi x}{L}\, dx$$

Split up the integral on the left and bring the constants in front.

$$\sum_{n=1}^{\infty} \left[B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t)\right] \int_0^L \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}\, dx = \int_0^L g(x) \sin \frac{p\pi x}{L}\, dx$$

Because the sine functions are orthogonal, this integral on the left is zero if $n \neq p$. Only if $n = p$ is it nonzero.

$$\left[B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t)\right] \int_0^L \sin^2 \frac{n\pi x}{L}\, dx = \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

Evaluate the integral on the left.

$$\left[B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t)\right] \frac{L}{2} = \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

Multiply both sides by $2/L$.

$$B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t) = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

This is a first-order linear inhomogeneous ODE, so it can be solved with an integrating factor I.

$$I = \exp\left(\int^t \frac{kn^2\pi^2}{L^2}\, dx\right) = \exp\left(\frac{kn^2\pi^2}{L^2} t\right)$$

Multiply both sides of the ODE by I.

$$\exp\left(\frac{kn^2\pi^2}{L^2} t\right) B_n'(t) + \frac{kn^2\pi^2}{L^2} \exp\left(\frac{kn^2\pi^2}{L^2} t\right) B_n(t) = \left[\frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx\right] \exp\left(\frac{kn^2\pi^2}{L^2} t\right)$$

The left side can be written as $d/dt(IB_n)$ by the product rule.

$$\frac{d}{dt}\left[\exp\left(\frac{kn^2\pi^2}{L^2} t\right) B_n(t)\right] = \left[\frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx\right] \exp\left(\frac{kn^2\pi^2}{L^2} t\right)$$

Integrate both sides with respect to t.

$$\begin{aligned}\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n(t) &= \int^t \left[\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + C \\ &= \left[\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right]\int^t \exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + C \\ &= \left[\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right]\frac{L^2}{kn^2\pi^2}\exp\left(\frac{kn^2\pi^2}{L^2}t\right) + C \\ &= \left[\frac{2L}{kn^2\pi^2}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}t\right) + C\end{aligned}$$

Solve for $B_n(t)$.

$$B_n(t) = \left[\frac{2L}{kn^2\pi^2}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right] + C\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)$$

Set $t = 0$ and use the formula found in the beginning for $B_n(0)$ to determine C.

$$B_n(0) = \left[\frac{2L}{kn^2\pi^2}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right] + C = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$C = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx - \left[\frac{2L}{kn^2\pi^2}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right]$$

Therefore,

$$\begin{aligned}B_n(t) &= \left[\frac{2L}{kn^2\pi^2}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right] + \left\{\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx - \left[\frac{2L}{kn^2\pi^2}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right]\right\}\exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \\ &= \left[\frac{2L}{kn^2\pi^2}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right]\left[1-\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\right] + \left[\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right),\end{aligned}$$

and the solution to the PDE is

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi x}{L} \\ &= \sum_{n=1}^{\infty}\left\{\left[\frac{2L}{kn^2\pi^2}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx\right]\left[1-\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\right]\right. \\ &\qquad\left. + \left[\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\right\}\sin\frac{n\pi x}{L}.\end{aligned}$$